References

[1] Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1960.

[2] Stewartson, K., *The Theory of Laminar Boundary Layers in Compressible Fluids*, Oxford Mathematical Monographs, Oxford University Press, Oxford, England, 1964.

[3] Liu, W. S., "The Analysis of Shock Structure and Nonequilibrium Boundary-Layer Induced by a Normal Shock Wave in an Ionized Argon Flow," University of Toronto, Canada, UTIAS Rept. 198, 1975.

[4] Liu, W. S., Whitten, B. T., and Glass, I. I., "Ionizing Argon Boundary Layer. Part 1. Quasi-Steady Flat-Plate Laminar Boundary-Layer Flows," *Journal of Fluid Mechanics*, Vol. 87, 1978, Pt. 4, pp. 609-640.

[5] Yakhot, A., Ben-Dor, G., Rakib, Z., and Igra, O., "A New Approach to the Solution of the Boundary Layer Equations of an Ideal Compressible Flow over a Flat Plate," *The Aeronautical Journal*, Vol. 85, 1981, Paper 845, pp. 34-35.

[6] Töpfer, C., "Bemerkungen zu dem Aufsatz von H. Blasius," *Zeitschr. f. Math. a. Phys.*, Vol. 60, 1912, pp. 397-398.

[7] Ben-Dor, G., Rakib, Z., and Igra, O., "The Boundary Layer of a Compressible Singly Ionized Frozen Flow over a Flate Plate—A New Method of Solution," Collection of Papers of the 25th Israel Annual Conference on Aviation and Astronautics, 1983.

Calculation of Three-Dimensional Instability of a Blasius Boundary Layer

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Introduction

TRANSITION in a boundary layer below a freestream of low turbulence level and small spatial variations consists of the following stages: 1) the appearance of two-dimensional Tollmien-Schlichting waves; 2) with the increase of the Reynolds number as flow goes downstream, three-dimensional waves appear—the subject of the present Note; 3) the onset of nonlinear effects as the wave amplitude grows; and 4) the onset of randomness beginning as modulation of the higher harmonics in the concentrated shear layers.

Stage 1 can be well described by linear theory for two-dimensional infinitesimal disturbances. The third stage is distinguished by the onset and increase of the nonlinear effects. Perhaps the problems involved in this state are the most important and complicated in the transition process.

Wortmann[1] proposed a model for the disturbance structure in the second, incipiently three-dimensional stage, based on his own measurements. The disturbance was proposed to consist of longitudinal counterrotating sheets of vorticity, inclined downstream and overlapping each other like roof shingles. Complete three-dimensional solutions, even of linearize equations, would be very time-consuming. In this Note we show that Squire's theorem[2] relating two and three-dimensional disturbances, together with the fact that (longitudinal) wave number α appears in the Orr-Sommerfeld linearized stability equation only in the group αR (where R is the Reynolds number), can be used to relate known two-dimensional eigenfunctions to a small three-dimensional disturbance with the same αR, but with a *resultant* wave number $\sqrt{(\alpha^2+\beta^2)}$ equal to the two-dimensional α. We consider only neutral disturbances for simplicity, but this is not a necessary restriction.

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Small-Disturbance Theory

Following Benney[3] we assume that the small three-dimensional disturbance has the form

$$u=\hat{u}(y)\,e^{i\alpha(x-ct)}\cos\beta z \tag{1a}$$

$$v=\hat{v}(y)\,e^{i\alpha(x-ct)}\cos\beta z \tag{1b}$$

$$w=\hat{w}(y)\,e^{i\alpha(x-ct)}\sin\beta z \tag{1c}$$

$$p=\hat{p}(y)\,e^{i\alpha(x-ct)}\cos\beta z \tag{1d}$$

where the y-dependent amplitudes indicated by a caret above the quantity are the "disturbance wave functions" and the $\sin\beta z$ factor in w is necessary to satisfy the continuity equation. Substituting into the Navier-Stokes equations and linearizing, we obtain a system of equations for the disturbance wave functions, on a two-dimensional mean flow $U(y)$, as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\gamma^2\right)^2\hat{v}=i\alpha R\left\{(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\gamma^2\right)\hat{v}-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\hat{v}\right\} \tag{2}$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\gamma^2-i\alpha R(U-c)\right\}\hat{\omega}_2=-\beta R\frac{\mathrm{d}U}{\mathrm{d}y}\hat{v} \tag{3}$$

$$\gamma^2\hat{u}=i\alpha\frac{\mathrm{d}\hat{v}}{\mathrm{d}y}-\beta\hat{\omega}_2 \tag{4a}$$

$$\gamma^2\hat{w}=i\alpha\hat{\omega}_2-\beta\frac{\mathrm{d}\hat{v}}{\mathrm{d}y} \tag{4b}$$

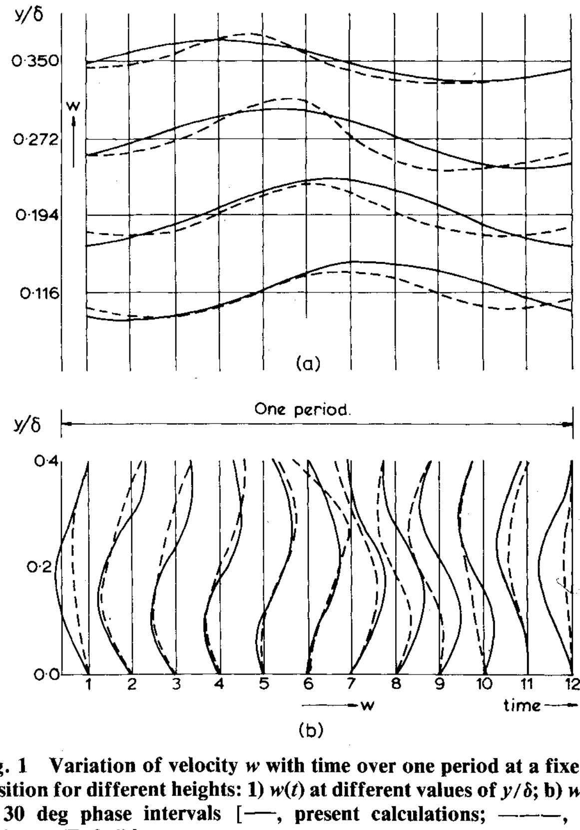

Fig. 1 Variation of velocity w with time over one period at a fixed x position for different heights: 1) $w(t)$ at different values of y/δ; b) $w(y)$ at 30 deg phase intervals [—, present calculations; – – –, experiment (Ref. 1)].

$$\gamma^2 \hat{\omega}_1 = i\alpha \frac{d\hat{\omega}_2}{dy} - \beta\left(\frac{d^2}{dy^2} - \gamma^2\right)\hat{v} \tag{4c}$$

$$\gamma^2 \hat{\omega}_3 = \beta \frac{d\hat{\omega}_2}{dy} - i\alpha\left(\frac{d^2}{dy^2} - \gamma^2\right)\hat{v} \tag{4d}$$

where $\gamma^2 = \alpha^2 + \beta^2$ and $\hat{\omega}_1$, $\hat{\omega}_2$, and $\hat{\omega}_3$ are disturbance vorticity functions for the x, y, and z components, respectively, defined by

$$\hat{\omega}_1 = \hat{m}_1(y) e^{i\alpha(x-ct)} \sin\beta z$$

$$\hat{\omega}_2 = \hat{m}_2(y) e^{i\alpha(x-ct)} \sin\beta z$$

$$\hat{\omega}_3 = \hat{m}_3(y) e^{i\alpha(x-ct)} \cos\beta z$$

Benney applied this system of equations to a simplified mean velocity profile, but here we use the numerically exact Blasius profile.

Equation (2) is just the two-dimensional Orr-Sommerfeld equation for the phase velocity eigenvalue $c = c(\alpha R, \gamma)$ and the corresponding eigenfunction $\hat{v}(y)$. We will subscribe the eigenvalues "2D" and "3D" according to whether the equation is applied to two- or the three-dimensional disturbances: γ^2 is either α_{2D}^2 or $\alpha_{3D}^2 + \beta^2$.

Squire's theorem implies that the two-dimensional eigenfunction of Eq. (2) is proportional to the three-dimensional one if and only if

$$c_{2D} = c_{3D} \tag{5a}$$

$$\alpha_{2D} = \gamma \tag{5b}$$

$$\alpha_{2D} R_{2D} = \alpha_{3D} R_{3D} \tag{5c}$$

Given solutions of Eq. (2) for $\hat{v}$ from previous wholly two-dimensional calculations, one can integrate Eq. (3) to obtain $\hat{\omega}_2$ and then obtain all of the three-dimensional eigenfunctions from Eq. (4).

Numerical Integration and Examples

Equation (3) has to be solved subject to the two-point boundary conditions $\hat{\omega}_2(y) = 0$ for $y/\delta = 0$ and for $y/\delta = \infty$ (say $\eta = 9$). Although Eq. (3) is only a second-order ordinary differential equation for a complex variable, it shares some numerical difficulties with the fourth-order Orr-Sommerfeld equation because both become "stiff" at high Reynolds numbers. After failing to solve Eq. (3) with a conventional shooting method, we used the Keller box method[4] with complete success.

We have used the existing results of Radbill and McCue[5] for the two-dimensional eigenfunction: these results correspond to the data of Schubauer and Skramstad[6] on the lower branch of the neutral curve where $Re_{\delta^*} = 897.93$, $\alpha\delta^* = 0.18182$, $c_r/U_e = 0.33156$, and $c_i = 0$. To compare with the experimental results of Wortmann[1] we have chosen the wavelengths in the x and z directions to be the same as in his experiment, i.e., $\lambda_x = 18.5$ cm, $\lambda_z = 29$ cm, so

$$\beta/\alpha_{3D} = \lambda_x/\lambda_z = 18.5/29$$

According to Eq. (6) we get

$$\gamma\delta^* = \alpha_{2D}\delta^* = 0.18182$$

$$\alpha_{3D}\delta^* = \gamma\delta^*/\sqrt{1 + (\beta/\alpha_{3D})^2} = 0.15329$$

$$\beta\delta^* = 0.09779$$

$$R_{3D\delta^*} = (\alpha_{2D} R_{2D\delta^*})/\alpha_{3D} = 1065.1$$

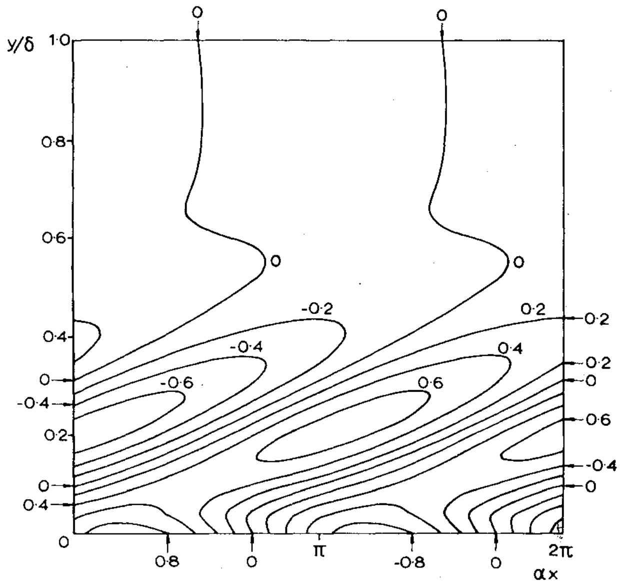

Fig. 2 Contour map of calculated longitudinal vorticity (variation with t replotted against x for clarity).

Details of the three-dimensional eigenfunctions are given in Ref. 7. Here we present only a small selection for comparison with the data of Wortmann. It is worth noting that our results for the spanwise velocity w (the simplest characteristic of a three-dimensional disturbance) agree well with those of Benney[3] who assumed a piecewise-linear $U(y)$ and took values of α, β, and R considerably different from ours.

Although we have chosen the same α and β as in Wortmann's experiments, there are some differences here too. For example, in the calculation, the boundary layer is Blasius, the disturbances are neutral, the dimensionless frequency $F(=\omega\nu/U_\infty^2) = 0.477 \times 10^{-4}$ and $R_{\delta^*} = 1065$. In the experiment, there was an adverse pressure gradient corresponding to a Falkner-Skan β of about -0.12, the disturbances were strongly amplified ($F = 1.2 \times 10^{-2}$), and $R_{\delta^*} = 1230$ (the least important difference). However, the correspondence of our w results and Benney's calculation suggest that our w should be comparable to Wortmann's.

Figure 1 shows the variation of velocity w with height and time over one period, the experimental results being interpolated where necessary. The w scale is, of course, immaterial to a comparison with linearized calculations: the rms amplitude in the experiments was about 2.6% of the freestream speed.

One of the important features of three-dimensional disturbances is that, at a fixed x position, the velocity w varies periodically with time t not only in its magnitude but also in the value of y at which $w = 0$. Our calculations have reproduced this feature (Fig. 1b) and the general trends of calculation and experiment are in good agreement. Figure 1a shows that the phase shift with height has also been well reproduced.

Based on his observations, Wortmann proposed a "roof shingle" model, as mentioned above. To test this idea, we have drawn a contour map for longitudinal vorticity in Fig. 2. This picture is compatible with the presence of overlapping longitudinal vortices. It is especially noteworthy that the detailed calculations show the longitudinal component of vorticity to be the largest of the three-components even though the "sweepback angle" of the disturbance, $\tan^{-1}(\lambda_x/\lambda_z)$ is only 33 deg.

Conclusions

A method has been developed to permit the extension of two-dimensional eigenfunction results to three-dimensional cases.

Some important features of three-dimensional disturbances and the "roof shingle" shape proposed by Wortmann[1] have been reproduced by the present calculation method. This fact shows that the eigenfunctions obtained from linear theory can be used in some extent to describe three-dimensional disturbance structure.

Acknowledgments

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References

[1]Wortmann, F. X., "The Incompressible Fluid Motion Downstream of Two-Dimensional Tollmien-Schlichting Waves," AGARD CP 224, 1977, p. 12.1.

[2]Squire, H. B., "On the Stability of the Three-Dimensional Disturbances of Viscous Flow between Parallel Walls," *Proceedings of the Royal Society*, Ser. A., Vol. 142, 1933, p. 621 (see also Rosenhead, L., *Laminar Boundary Layers,* Clarendon Press, Oxford England, 1963, pp. 514-515).

[3]Benney, D. J., "Finite Amplitude Effects in an Unstable Laminar Boundary Layer," *Physics of Fluids*, Vol. 7, 1964, p. 319.

[4]Cebeci, T. and Bradshaw, P., *Momentum Transfer in Boundary Layers,* McGraw-Hill Book Co., New York, 1977, pp. 213-230.

[5]Radbill, J. R. and McCue, G. A., *Quasilinearization and Nonlinear Problems in Fluid and Orbital Mechanics*, American Elsevier, New York, 1970, p. 73.

[6]Schubauer, G. B. and Skramstad, H. K., "Laminar Boundary-Layer Oscillations and Transition on a Flat Plate," *Journal of the Aeronautical Sciences*, Vol. 14, Feb. 1947, p. 69.

[7]Chen, M. Z. and Bradshaw, P., "Calculation of Three-Dimensional Unstability Modes in a Blasius Boundary Layer," Imperial College, London, Aero Rept. 81-01, 1981.

Contribution to the Reynolds Stress Model as Applied to Near-Wall Region

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I. Introduction

SOME complicated turbulent flowfields have recently been solved by using the Reynolds stress (RS) model. This indicates that the RS model has become a practical approach which can be used to satisfy technical demands. However, with the exception of Hanjalic and Launder,[1] there have been few authors who have solved the whole flowfield, including the near wall region. Here, it is emphasized that the near wall region used in this Note includes the solution up to the wall.

Since the RS model usually used[1-7] is based on the considerations of isotropic turbulent flow, it can not be applied directly to a strongly anisotropic flow, such as a flow near a wall. Hence, in most reports,[1-7] the near wall region is excluded from calculations. The wall boundary conditions are applied in the proximity of the wall and the flow properties are matched, for instance, to the law of the wall.

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Hanjalic and Launder (HL) succeeded in solving the whole flowfield including the near wall region by estimating the direct viscous effects on various transport processes and introducing a function to correct the profile of the Reynolds shear stress near the wall. The HL model is simplified by using two experimental approximations and includes only the transport equation of the Reynolds shear stress, $\overline{uv}$.

For engineers, it is desirable to solve general three-dimensional flow by a complete RS model. There has been no attempt to solve the whole flowfield up to a wall with this approach.

This Note presents a model for solving the whole flowfield including the near wall region by using a complete RS model. The present method is based on the idea given in the next section, in addition to Hanjalic and Launder's considerations about the direct viscous effects. The newly developed model was applied to two-dimensional turbulent flows. The results showed good agreement with the experimental data in the whole flowfield including the near wall region.

II. Development of Model

The pressure-strain correlation term arising in the RS model is also called the redistribution term of turbulent energy and has a characteristic of making a flow isotropic. From this characteristic, if the function of this term is restrained in a near wall region, it is anticipated that the model can describe the anisotropic flow in that region without serious error.

Using this idea, functions multiplying the pressure-strain terms were introduced in the transport equations of Reynolds stress. The transport equations for two-dimensional shear flow are, then, written as follows.

$$\frac{D\overline{u_iu_j}}{Dt} = -\left(\overline{u_ju_k}\frac{\partial U_i}{\partial x_k} + \overline{u_iu_k}\frac{\partial U_j}{\partial x_k}\right) + \nu\frac{\partial^2 \overline{u_iu_j}}{\partial x_k^2} + \pi_{ij}F$$
$$+ C\frac{\partial}{\partial x_k}\left[\frac{k}{\epsilon}\left(\overline{u_iu_\ell}\frac{\partial \overline{u_ju_k}}{\partial x_\ell} + \overline{u_ju_\ell}\frac{\partial \overline{u_ku_i}}{\partial x_\ell} + \overline{u_ku_\ell}\frac{\partial \overline{u_iu_j}}{\partial x_\ell}\right)\right]$$
$$-2\left(\frac{1-f_s}{3}\delta_{ij} + \frac{\overline{u_iu_j}}{2}\frac{f_s}{k}\right)\epsilon$$

where π_{ij} is the pressure-strain correlation term which is the same one as Hanjalic and Launder used. F represents the introduced functions, which were determined by trial and error, and f_s is the function proposed by Hanjalic and Launder to deal with the viscous effects of the dissipation ϵ.

The following three-F functions were used in each transport equation of Reynolds stress.

For $\overline{uv}$:

$$F1 = \exp\{-0.4/[1+(R_t/50)^2]\}$$

For $\overline{u^2}, \overline{v^2}$:

$$F2 = \exp[-1.2/(1+R_t/200)]$$
$$= 0.31 - 0.12(5-R_t)^{0.5} \qquad R_t < 5$$

For $\overline{w^2}$:

$$F3 = \exp[-1.2/(1+R_t/50)]$$
$$= 0.336 - 0.13(5-R_t)^{0.5} \qquad R_t < 5$$

where R_t is the turbulent Reynolds number, $R_t = k^2/\nu\epsilon$.

Here, it must be emphasized that these functions are significant as a combination and must not be considered separately. These functions are one of the combinations able to produce reasonable results in the near wall region; other